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STATIONARY TRANSFER IN FIBROUS COMPOSITE MATERIALS

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The effective heat-conduction coefficients are calculated for fibrous materials of different structure.

The general methods of investigating transfer processes in heterogeneous systems developed in [1, 2] in application to dispersed media with spherical particles can be used successfully also in describing these processes in materials of a different structure. Stationary transfer in fibrous materials, which is used extensively in engineering and being of considerable applied interest, is examined below in the example of the heat-conduction process.

In the general case, materials consisting of a continuous medium and fibers with different physical properties distributed therein are not isotropic, where the nature and degree of the anisotropy are determined by the fiber packing features. Taking into account that the mean heat flux $\mathbf{q}$ and the mean temperature gradient τ are real vectors, we see that the quantity Λ must be considered a real tensor of the second rank in the linear relationship

$$\mathbf{q} = -\Lambda\nabla\tau, \tag{1}$$

replacing the Fourier law in the case under consideration. The effective heat-conduction coefficients can comprise a nonglobal tensor in other dispersed media also, e.g., in media with spheroidal particles having a preferred direction of orientation of their axes of symmetry [3]. Equation (1) can be obtained strictly by taking the average of the local Fourier relationships, which are valid within and outside the fibers in either small physical volume (in this case the linear scale of the quantities $\mathbf{q}$ and $\boldsymbol{\tau}$ should considerably exceed the internal structural scale of the material) as was done in [1], or in the ensemble of admissible fiber configurations analogous in meaning to the ensemble of configurations of systems of rigid spheres studied in [2].

Let us investigate a material with extended parallel fibers first. The cross section of each fiber is a circle of radius a; the centers of such circles are arranged randomly

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in a plane normal to the fiber axes, i.e., can occupy different points of this plane with equal probability. The axes of a Cartesian coordinate system, one of which is directed along the fiber axis while the other two are arbitrarily arranged in the plane mentioned, are the principal axes of the heat-conduction tensor Λ in which it is a diagonal. One of the eigenvalues of Λ hence equals the coefficient of "longitudinal" heat conduction

$$\lambda^{(1)} = \lambda_{\parallel} = \varepsilon\lambda_0 + \rho\lambda_1, \quad \varepsilon = 1 - \rho, \tag{2}$$

and the two others agree with the coefficient of "transverse" heat conduction, i.e.,

$$\lambda^{(2)} = \lambda^{(3)} = \lambda_{\perp}, \tag{3}$$

where $\lambda_{\perp}$ can be calculated exactly as the coefficient of effective heat conduction of the isotropic dispersed medium with spherical particles considered in [4]. The sole difference from the analysis in [4], where configurations of systems of spheres in a volume were examined, is that in this case it is necessary to consider configurations of circles which are the fiber cross sections in a plane perpendicular to their axes.

Omitting the details of the discussion, which is perfectly analogous to that in [2, 4], we arrive at the formula

$$\lambda_{\perp} = \lambda_0\beta, \ \beta\mathbf{E} = \mathbf{E} + (\varkappa - 1)\frac{\rho}{\pi a^2}\int\limits_{x=a}\tau^*(\mathbf{x}+\mathbf{r}\,|\,\mathbf{r})\,\mathbf{n}dx, \ \varkappa = \frac{\lambda_1}{\lambda_0}, \tag{4}$$

where the integration is over the contour of an isolated ("test") circle with center at the point $\mathbf{r}$, of the mean temperature τ^* on this contour. This is determined from the solution of the following plane problem on the temperature distribution in the neighborhood of the test fiber, obtained analogously to the spatial problem of the temperature distribution near a test spherical particle in [4]:

$$\begin{gathered}\nabla[\mathrm{B}(\xi)\nabla\tau'] = 0, \ x > a; \ \nabla\tau^* = 0, \ a > x \geqslant 0; \ \xi = x/a, \\ \tau' \to 0, \ x \to \infty; \ \tau^* < \infty, \ x = 0, \\ \tau' + \mathbf{E}\mathbf{x} = \tau^*, \ \lambda_0\mathbf{n}\nabla\tau' + \lambda_{\perp}\mathbf{n}\mathbf{E} = \lambda_{\perp}\mathbf{n}\nabla\tau^*, \ x = a.\end{gathered} \tag{5}$$

Here $\mathbf{Ex}$ is the mean temperature in the material, not perturbed by the test fiber (without limiting the generality, it can be considered that the vector $\mathbf{E}$ is in the plane of the problem), and τ' is the corresponding perturbation of the temperature field.

The test fiber can therefore be formally considered as submerged in a fictitious homogeneous medium whose heat conduction $\lambda_0 B(\xi)$ depends on the distance to the fiber surface. The analogy to the situation in [2, 4] is completely evident here. The expression for $B(\xi)$ has the same form as the expression for β in (4) if the volume fiber concentration ρ in the latter is replaced by the local concentration $\rho(\xi) = \rho\sigma(\xi)$, which is the probability of an event that the point at a distance $x = \xi a$ from the axis of the test fiber will be within the fiber. For materials with the random fiber packing assumed, $\sigma(\xi)$ is easily calculated from a consideration of the simple geometric problem: if there are two circles of radius a and $2a$ separated by the spacing x between their centers, then $\sigma(\xi)$ equals the ratio between the area of the smaller circle lying outside the larger circle to the total area of the small circle. We have

$$\sigma(\xi) = 1 - \frac{1}{\pi}\left\{\arccos\frac{\xi^2-3}{2\xi} + \arccos\frac{\xi^2+3}{4\xi} - \frac{1}{2}[(\xi^2-1)(9-\xi^2)]^{1/2}\right\} \tag{6}$$

for $1 < \xi < 3$ and $\sigma(\xi) = 1$ for $\xi > 3$. The difference of $\sigma(\xi)$ from one near the test fiber surface reflects the property of nonoverlapability of the fibers: the axes of two adjacent fibers cannot be closer than the spacing $2a$.

It is easy to see that the solution of (5) can be represented as

$$\tau' = f(\xi)\,\mathbf{Ex}, \quad \tau^* = \nu\mathbf{Ex}, \tag{7}$$

so that after evaluating the integral in (4) we obtain for β and $B(\xi)$

$$\beta = 1 + (\varkappa - 1)\rho\nu, \ \mathrm{B}(\xi) = 1 + (\varkappa - 1)\rho\sigma(\xi)\nu. \tag{8}$$

The problem permitting ν and $f(\xi)$ to be found is obtained easily after substituting (7) into (5), and has the form

$$\xi \frac{d^2 f}{d\xi^2} + 3 \frac{df}{d\xi} + \frac{d \ln B(\xi)}{d\xi} \left(\xi \frac{df}{d\xi} + f \right) = 0, \tag{9}$$

$$f = \nu - 1, \; df/d\xi = \varepsilon(\varkappa - 1)\nu, \; \xi = 1; \; f \to 0, \; \xi \to \infty$$

Let us note that the solution of the problem of a test spherical particle formulated in [4] can be also represented in the form (7), where a problem is obtained for ν and $f(\xi)$ which also has the form (9) but with another dependence $\sigma(\xi)$ [5].

For quite small ρ the effect of nonoverlappability of the fibers can be neglected, i.e., $\sigma(\xi) = 1$ and $B(\xi) = \beta$ can simply be taken, which corresponds to a representation of the test fiber submerged in a homogeneous fictitious medium whose properties agree with the mean effective properties of the material as a whole. In this case, we have the following approximate formula from the solution (5) or (9):

$$\beta = \frac{1}{2} \{(1 - 2\rho)(1 - \varkappa) + [(1 - 2\rho)^2 (1 - \varkappa)^2 + 4\varkappa]^{1/2}\}. \tag{10}$$

The approximation

$$\sigma(\xi) = \begin{cases} 0, & 1 < \xi < 2 \\ 1, & \xi > 2, \end{cases} \tag{11}$$

corresponding to the representation of a test fiber separated from a homogeneous medium by a concentric layer of thickness a in which the heat conduction agrees with that for the pure continuous phase of the material (see [2, 4]), is possible for concentrated material. In this case we obtain after simple calculations

$$\beta = [3\varkappa(1 - \rho) + 5 + 3\rho]^{-1} \{-[\varkappa(3 - 7\rho) + 1 + 7\rho] + \{[\varkappa(3 - 7\rho) + 1 + 7\rho]^2 + [3\varkappa(1 - \rho) + 5 + 3\rho][\varkappa(9 + 7\rho) + 7(1 - \rho)]\}^{1/2}\}. \tag{12}$$

Approximate formulas (10) and (12) are compared in Fig. 1 with the result from a numerical solution of the problem (9), and with the analog of the known Maxwell formula with the form

$$\frac{\beta - 1}{\beta + 1} = \frac{\varkappa - 1}{\varkappa + 1} \rho. \tag{13}$$

For values of $\varkappa$, small and commensurate to one, the approximate formula (12) yields a good result; for very large $\varkappa$ there is a substantial discrepancy between (12) and the exact result obtained from (8) with ν determined from the numerical solution of (9). These deductions are valid even for the dispersed medium investigated in [4, 5]. However, in contrast to the situation in [4, 5], the domain for validity of (10) turns out to be quite narrow; this formula is considerably less successful than even (13). Hence, the physically completely conceivable deduction follows that the effect of nonoverlappability for the fibers is relatively more substantial than for the particles.

Dependences of the relative transverse heat conductivity β on the concentration ρ are shown in Fig. 2 for different $\varkappa$, obtained as a result of numerical integration of problem (9).

Now, let us examine fibrous materials of more complex structure. The components of the heat-conduction tensor for such materials can be calculated, in principle, by means of the known quantities $\lambda_{\parallel}$ and $\lambda_{\perp}$, defined above, and by means of certain data about the structure

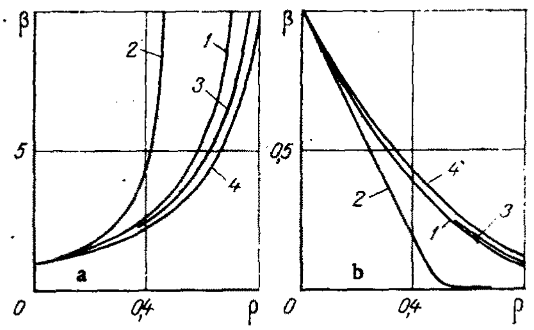

Fig. 1. Comparison between formulas for the relative transverse heat conduction of a material with parallel fibers for $\varkappa > 1$ (a) and $\varkappa < 1$ (b); 1) result obtained by numerical integration of the problem (9), 2) (10); 3) (12); 4) (13); for a) $\varkappa \approx \infty$; for b) $\varkappa \approx 0$.

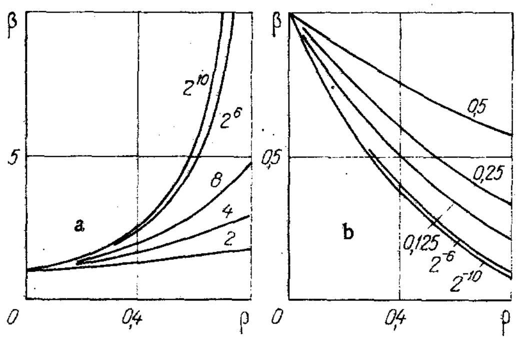

Fig. 2. Dependences of β on ρ for $\varkappa > 1$ (a) and $\varkappa < 1$ (b); the digits at the curves are values of $\varkappa$.

of the material. Let us first examine a material with elongated fibers whose axes are parallel to a certain plane (x, y), where the directions of the axes are described by a certain distribution function $F(\varphi)$, normalized to one, where φ is an angle measured in the usual manner from the x axis. Such material can be constructed by the random superposition of plane arrays with parallel fibers on each other, where the frequency of appearance of arrays with fibers of a given direction is controlled by the function $F(\varphi)$. Firstly, it is clear that for such material

$$\lambda^{(3)} = \lambda_{33} = \lambda_{\perp},\ \lambda_{i3} = \lambda_{3i} = 0\ \ (i \neq 3). \tag{14}$$

The remaining components of the heat-conduction tensor are easily evaluated by using the rule for transforming tensors with the change in the coordinate system. Thus, the system of coordinate axes being used here can be considered as obtained from the principal system of axes for arrays of parallel fibers oriented in the direction φ because of a unitary transformation, rotation through an angle φ, around the z axis. It is hence easy to find the components of the tensor Λ for this array, and to calculate such components even for the material under investigation as a whole by using the distribution function $F(\varphi)$ after taking the average. Consequently, the following formulas are obtained*

$$\begin{aligned}\lambda_{11} &= \int_0^{\pi} (\lambda_{\parallel}\cos^2\varphi + \lambda_{\perp}\sin^2\varphi)\,F(\varphi)\,d\varphi,\\ \lambda_{22} &= \int_0^{\pi} (\lambda_{\parallel}\sin^2\varphi + \lambda_{\perp}\cos^2\varphi)\,F(\varphi)\,d\varphi,\\ \lambda_{12} = \lambda_{21} &= (\lambda_{\perp} - \lambda_{\parallel})\int_0^{\pi} \cos\varphi\,\sin\varphi F(\varphi)\,d\varphi.\end{aligned} \tag{15}$$

In particular, for an equally probable distribution of directions of the fiber axes in the plane (x, y), we have $F(\varphi) = \pi^{-1}$ and

$$\lambda_{11} = \lambda^{(1)} = \lambda_{22} = \lambda^{(2)} = \lambda' = \frac{1}{2}(\lambda_{\parallel} + \lambda_{\perp}),\ \lambda_{12} = \lambda_{21} = 0. \tag{16}$$

This latter result can also be obtained more simply if it is taken into account that for equally probable values of φ we always have $\lambda_{11} = \lambda_{22}$, $\lambda_{12} = \lambda_{21} = 0$, i.e., the trace of the tensor Λ equals $2\lambda^{(1)} + \lambda^{(3)} = 2\lambda_{11} + \lambda_{\perp}$. On the other hand, the trace of the tensor is invariant relative to unitary transformations of coordinates, i.e., should equal $\lambda_{\parallel} + 2\lambda_{\perp}$ as before. Hence, (16) again follows at once.

It is easy to see that results (15) and (16) are true for materials with unelongated fibers (whose axes are curves of lines parallel to the (x, y) plane). Indeed, for this it

*Let us emphasize that the specific expressions and numerical results for the transverse heat-conduction coefficient of a material with elongated parallel fibers have been obtained above under the assumption of random packing of the fibers. The packing is somewhat ordered in plane arrays with parallel fibers, i.e., is distinct from random. Hence (15) and (16) must be considered approximate, true to the accuracy of possible fluctuations due to the packing details, for materials comprised on the arrays mentioned. Topological singularities of the materials with interwoven arbitrarily cambered fibers all differ analogously from those for materials with elongated fibers. Hence, (16) and (17) are also true in the general case but only approximately when applied to material of that kind; their errors, which can be considered moderate from general considerations, can finally be determined for different values of the parameters only by comparison with experiment results.

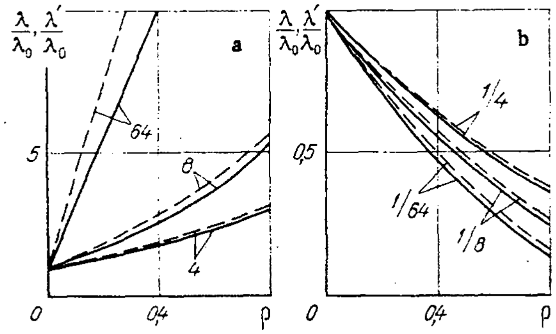

Fig. 3. Dependences of the relative heat conductivities λ/λ_0 (solid lines) and λ'/λ_0 (dashes) on ρ for $\varkappa > 1$ (a) and $\varkappa < 1$ (b); numbers on the curves are the values of $\varkappa$.

is only necessary to consider that the function $F(\varphi)$ describes the distribution of the local directions of the fiber axes (e.g., within the limits of a small physical volume).

Now, let the fibers be oriented arbitrarily in space, where the distribution of the directions of their axes is described at any point by a certain distribution function in which the usual polar and azimuthal angles or even the Euler angles figure as arguments. The components of the tensor Λ for fibers of fixed orientation are again obtained by using the tensor transfromation rule, and the components of this tensor (the effective heat conduction coefficients) for the material as a whole by using additional averaging utilizing the distribution function mentioned. The appropriate formal representations for the quantities λ_{ij} (i, j = 1, 2, 3) are easily found completely analogously to the representations (15); they are not written down here because of their awkwardness.

In the simplest case of a spatially isotropic material with randomly twisted and interwoven fibers, the tensor Λ is global. The heat-conduction coefficient of such a material is obtained easily from the condition of invariance of the trace of the tensor relative to rigid rotations of the coordinate axes, as mentioned earlier. We have

$$\lambda^{(i)} = \lambda = \frac{1}{3}(\lambda_{\parallel} + 2\lambda_{\perp}), \quad i = 1, 2, 3. \tag{17}$$

Dependences of the relative heat-conduction coefficients λ'/λ_0 and λ/λ_0 from (16) and (17) on ρ and $\varkappa$ are illustrated by the curves in Fig. 3.

In conclusion, let us note that because of the analogy in the formulation of the mathematical problems, the results obtained above can be used in calculations of not only the heat-conduction coefficients, but also of the effective diffusion coefficients and the electrophysical parameters (coefficients of conduction, dielectric and magnetic constants) of various fibrous materials.

NOTATION

a, fiber radius; E, mean temperature gradient in a direction normal to the fibers; $F(\varphi)$, distribution function of the directed fibers; $f(\xi)$, function introduced in (7); q, mean heat flux; x, distance from the axis of the test fiber; β, relative transverse heat conduction of a material with parallel fibers; $B(\xi)$, function introduced in (5) and (8); $\varepsilon = 1 - \rho$; $\varkappa = \lambda_1/\lambda_0$; λ, λ', effective heat-conduction coefficients defined by (17) and (16), respectively; $\lambda_{\parallel}$ and $\lambda_{\perp}$, longitudinal and transverse heat conduction of a material with parallel fibers; λ_0 and λ_1, heat conduction of a continuous medium and of fiber material, respectively; ν, eigenvalue of problem (9), which figures in (7) and (8); $\xi = x/a$; ρ, volume fiber concentration in the material; $\sigma(\xi)$, function introduced in (8) and defined in (6) for materials with random fiber arrangement; τ, mean temperature; and φ, azimuth angle in the (x, y) plane.

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